Vektorrechnung im $\mathbb{R}^3$ – Beispiele

1)

a) $\begin{pmatrix}4\\-7\\1\end{pmatrix} \times \begin{pmatrix}2\\3\\-5\end{pmatrix} = \begin{pmatrix}35-3\\2+20\\12+14\end{pmatrix} = \underline{\begin{pmatrix}32\\22\\26\end{pmatrix}}$

b) $\begin{pmatrix}-5\\3\\2\end{pmatrix} \times \begin{pmatrix}2\\-4\\-3\end{pmatrix} = \begin{pmatrix}-9+8\\4-15\\20-6\end{pmatrix} = \underline{\begin{pmatrix}-1\\-11\\14\end{pmatrix}}$

2) $A_{\#} = \left\| \begin{pmatrix}3\\-1\\2\end{pmatrix} \times \begin{pmatrix}5\\7\\-2\end{pmatrix} \right\| = \left\| \begin{pmatrix}2-14\\10+6\\21+5\end{pmatrix} \right\| = \left\| \begin{pmatrix}-12\\16\\26\end{pmatrix} \right\| = \sqrt{12^2+16^2+26^2} \approx \underline{32{,}8\ \text{FE}}$

$A_{\Delta} = \frac{A_{\#}}{2} \approx \underline{16{,}4\ \text{FE}}$

3) $A(-3|1|4) \quad B(-1|0|5)$

a) $\overrightarrow{AB} = \begin{pmatrix}2\\-1\\1\end{pmatrix}$

$\underline{g:\ X = \begin{pmatrix}-3\\1\\4\end{pmatrix} + t\begin{pmatrix}2\\-1\\1\end{pmatrix}}$

b) $g: \begin{cases} x = -3+2t \\ y = 1-t \\ z = 4+t \end{cases} \quad |\cdot 2 \quad \oplus \quad \oplus$

$$\begin{aligned} x &= -3+2t \\ 2y &= 2-2t \\ \hline x+2y &= -1 \end{aligned} \quad \oplus \qquad\qquad \begin{aligned} y &= 1-t \\ z &= 4+t \\ \hline y+z &= 5 \end{aligned} \quad \oplus$$

$\underline{g: \begin{cases} x+2y = -1 \\ y+z = 5 \end{cases}}$

4) $A(1|2|3) \quad B(3|4|-1) \quad C(4|\ \ |3)$

$\overrightarrow{AB} = \begin{pmatrix}2\\2\\-4\end{pmatrix} \parallel \begin{pmatrix}1\\1\\-2\end{pmatrix} \Rightarrow g_{AB}: X = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}1\\1\\-2\end{pmatrix}$

$C \text{ in } g: \begin{pmatrix}4\\5\\3\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}1\\1\\-2\end{pmatrix} \qquad \begin{aligned} 4 &= 1+t \leadsto t = 3 \\ 5 &= 2+t \leadsto t = 3 \\ 3 &= 3-2t \leadsto t = 0 \end{aligned} \ \neq \ \Rightarrow \ \underline{\underline{C \notin g_{AB}}}$

Die 3 Punkte liegen nicht auf derselben Geraden.

5.

a) $g: X = \begin{pmatrix}0\\0\\1\end{pmatrix} + s\begin{pmatrix}3\\-4\\1\end{pmatrix}$ $\quad h: X = \begin{pmatrix}1\\-2\\3\end{pmatrix} + t\begin{pmatrix}-6\\8\\-2\end{pmatrix}$

$u \cdot \begin{pmatrix}3\\-4\\1\end{pmatrix} \stackrel{!}{=} \begin{pmatrix}-6\\8\\-2\end{pmatrix}$ $\quad \begin{matrix}3u=-6 \rightsquigarrow u=-2\\ -4u=8 \rightsquigarrow u=-2\\ u=-2\end{matrix} \Big\} = \Rightarrow \begin{pmatrix}3\\-4\\1\end{pmatrix} \parallel \begin{pmatrix}-6\\8\\-2\end{pmatrix} \Rightarrow \underline{g \parallel h}$

$g \equiv h$?

$\begin{pmatrix}1\\-2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} + s \cdot \begin{pmatrix}3\\-4\\1\end{pmatrix}$ $\quad \begin{matrix}1=3s \rightsquigarrow s=\frac{1}{3}\\ -2=-4s \rightsquigarrow s=\frac{1}{2}\\ 3=1+s \rightsquigarrow s=2\end{matrix} \Big\} \neq \Rightarrow H \notin g \Rightarrow \underline{g \not\equiv h}$

g und h sind parallel, aber nicht ident.

b) $g: X = \begin{pmatrix}-2\\3\\-1\end{pmatrix} + t\begin{pmatrix}1\\-1\\1\end{pmatrix}$ $\quad h: X = \begin{pmatrix}5\\8\\-4\end{pmatrix} + s\begin{pmatrix}4\\2\\1\end{pmatrix}$

$\begin{pmatrix}1\\-1\\1\end{pmatrix} \nparallel \begin{pmatrix}4\\2\\1\end{pmatrix} \Rightarrow g \nparallel h$

$g = h$: $\quad \begin{aligned} \text{I}&: -2 + t = 5 + 4s\\ \text{II}&: 3 - t = 8 + 2s\\ \text{III}&: -1 + t = -4 + s \end{aligned}$ $\Big\} \oplus$

$\text{I} + \text{II}: 1 = 13 + 6s \rightsquigarrow -12 = 6s \rightsquigarrow s = -2$

s in I: $-2 + t = 5 - 8 \rightsquigarrow t = -1$

s und t in III: $-1 - 1 = -4 - 2$

$-2 = -6$

f.A. $\Rightarrow$ g und h schneiden einander nicht $\Rightarrow$ <u>g u. h windschief</u>

c) $g: X = \begin{pmatrix}18\\-1\\14\end{pmatrix} + s\begin{pmatrix}-3\\1\\2\end{pmatrix}$ $\quad h: X = \begin{pmatrix}7\\0\\0\end{pmatrix} + t\begin{pmatrix}5\\-1\\2\end{pmatrix}$

$\begin{pmatrix}-3\\1\\2\end{pmatrix} \nparallel \begin{pmatrix}5\\-1\\2\end{pmatrix} \Rightarrow g \nparallel h$

$g = h$: $\quad \begin{aligned} \text{I}&: 18 - 3s = 7 + 5t\\ \text{II}&: -1 + s = -t\\ \text{III}&: 14 + 2s = 2t \end{aligned}$ $\quad |\cdot 3 \Big\} \oplus$ $\quad \begin{aligned} 18 - 3s &= 7 + 5t\\ -3 + 3s &= -3t\\ \hline 15 &= 7 + 2t \end{aligned} \Big\} \oplus \rightsquigarrow t = 4$

t in II: $-1 + s = -4 \rightsquigarrow s = -3$

s u. t in III: $14 - 6 = 8 \rightsquigarrow 8 = 8$

w.A. $\Rightarrow \underline{g \times h}$

s in g: $S = \begin{pmatrix}18\\-1\\14\end{pmatrix} - 3\begin{pmatrix}-3\\1\\2\end{pmatrix} = \begin{pmatrix}27\\-4\\8\end{pmatrix}$ $\quad \underline{\underline{S(27/-4/8)}}$

6, $A(4/0/3) \quad B(2/2/-1) \quad C(0/-3/5)$

$$\vec{AB} = \begin{pmatrix}-2\\2\\-4\end{pmatrix} \parallel \begin{pmatrix}-1\\1\\-2\end{pmatrix} \qquad \vec{AC} = \begin{pmatrix}-4\\-3\\2\end{pmatrix}$$

$$\underline{E:\ X = \begin{pmatrix}4\\0\\3\end{pmatrix} + s\cdot\begin{pmatrix}-1\\1\\-2\end{pmatrix} + t\cdot\begin{pmatrix}-4\\-3\\2\end{pmatrix}}$$

$$\vec{n} = \begin{pmatrix}-1\\1\\-2\end{pmatrix} \times \begin{pmatrix}-4\\-3\\2\end{pmatrix} = \begin{pmatrix}2-6\\8+2\\3+4\end{pmatrix} = \begin{pmatrix}-4\\10\\7\end{pmatrix}$$

$$\underline{E:\ \begin{pmatrix}-4\\10\\7\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-4\\10\\7\end{pmatrix}\begin{pmatrix}4\\0\\3\end{pmatrix}}$$

$$\underline{E:\ -4x+10y+7z=5}$$

7, $E_1:\ 2x-y+z=3 \qquad E_2:\ -x+3y+3z=1$

$$\vec{n}_1 = \begin{pmatrix}2\\-1\\1\end{pmatrix} \nparallel \begin{pmatrix}-1\\3\\3\end{pmatrix} = \vec{n}_2 \quad \Rightarrow \quad \underline{E_1 \times E_2}$$

$$\begin{aligned} \mathrm{I}:&\ 2x - y + z = 3 \\ \mathrm{II}:&\ -x + 3y + 3z = 1 \quad |\cdot 2 \ \rangle + \end{aligned}$$

$z = t$

$5y + 7z = 5 \leadsto y = 1 - \frac{7}{5}z \longrightarrow y = 1 - \frac{7}{5}t$

y in I: $2x - 1 + \frac{7}{5}z + z = 3 \leadsto x = 2 - \frac{6}{5}z \longrightarrow x = 2 - \frac{6}{5}t$

$$\underline{\underline{g:\ X = \begin{pmatrix}2\\1\\0\end{pmatrix} + t\cdot\begin{pmatrix}-\frac{6}{5}\\-\frac{7}{5}\\1\end{pmatrix}}}$$

8, $E:\ -x+3y-5z=3 \qquad g:\ X = \begin{pmatrix}4\\-1\\2\end{pmatrix} + t\cdot\begin{pmatrix}1\\4\\3\end{pmatrix}$

g in E: $-(4+t)+3(-1+4t)-5(2+3t)=3$

$$-4-t-3+12t-10-15t = 3$$

$$-4t = 20$$

$$t = -5$$

t in g: $S = \begin{pmatrix}4\\-1\\2\end{pmatrix} - 5\begin{pmatrix}1\\4\\3\end{pmatrix} = \begin{pmatrix}-1\\-21\\-13\end{pmatrix} \qquad \underline{\underline{S(-1/-21/-13)}}$